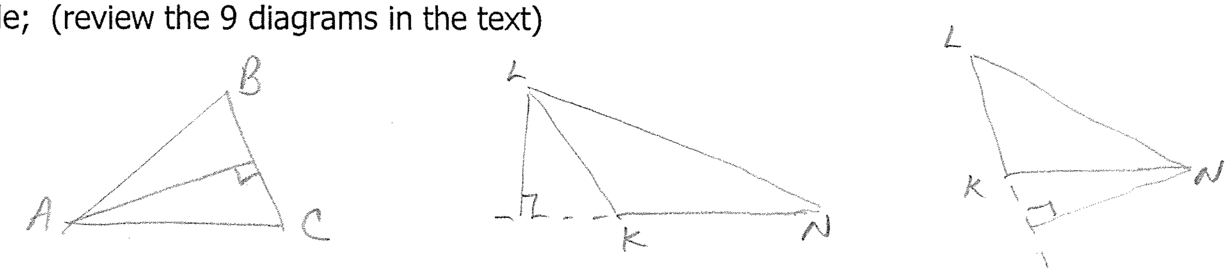
Geometry (H)
Section 4.7 – Altitudes, Medians, and the Equidistance Theorems

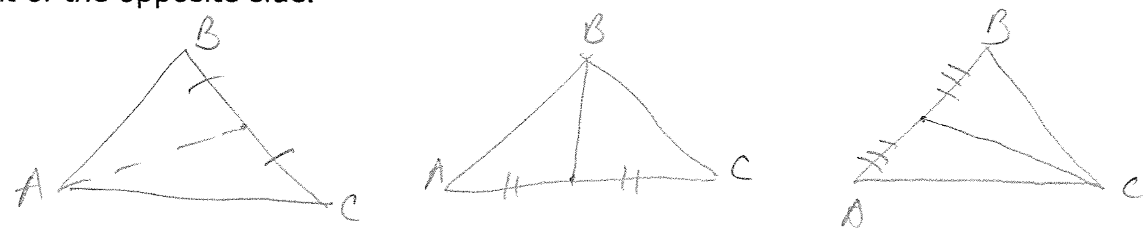
KEY for Notes

Definitions:

Altitude – in a triangle, it is the segment from a vertex perpendicular to the opposite side; (review the 9 diagrams in the text)

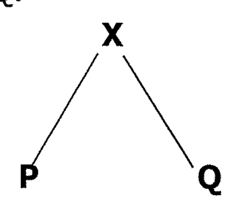


Median – a segment associated with triangles; a segment that connects a vertex to the midpoint of the opposite side.

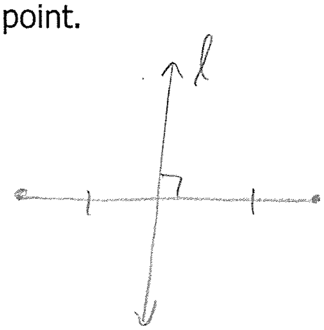


The **distance** between two objects is the length of the shortest path joining them. (Postulate: The shortest path between two points is the segment joining them.)

If two points P and Q are the same distance from a third point X, then X is said to be **equidistant** from P and Q.



Perpendicular Bisector of a segment – a line, ray, or segment that is perpendicular to the segment at its midpoint.

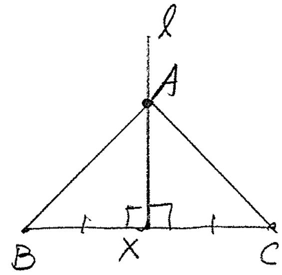


Theorems:

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of that segment.

Given: Line ℓ is the $\perp$ bisector of $\overline{BC}$
A is on ℓ.

Diagram:



Prove: $AB = AC$

1 Line ℓ is the $\perp$ bisector of $\overline{BC}$. → 2 $\angle AXC$ & $\angle AXB$ are right $\angle$s. → 3 $\angle AXC \cong \angle AXB$

1 → 4 $\overline{BX} \cong \overline{XC}$

5 $\overline{AX} \cong \overline{AX}$

3, 4, 5 → 6 $\triangle ABX \cong \triangle ACX$ → 7 $\overline{AB} \cong \overline{AC}$ → 8 $AB = AC$

1 Given
2 $\perp$ lines form Rt $\angle$s
3 All Rt. $\angle$s $\cong$.
4 Def. of seg. bisector
5 Reflexive prop
6 SAS $\cong$ SAS
7 CPCTC
8 If segmts $\cong$, their measures are =.

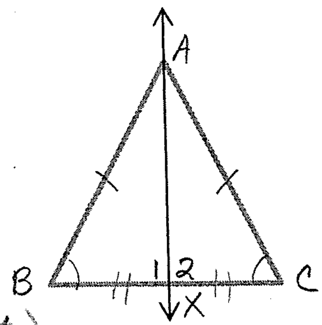
AC = AO

If a point is equidistant from the endpoints of a segment, then the point lies on the perpendicular bisector of the segment.

bisector of vertex ∠ of isos △ is ⊥ to base at midpt

Given: AB = AC

Diagram:



Prove: A is on the perpendicular bisector of $\overline{BC}$.

(Draw $\overline{AX}$ such that it goes thru midpt of BC)

(Postulate: A segment has one & only 1 midpt)

Part I

① Draw $\overline{AX}$ so that it goes thru midpt of $\overline{BC}$ → ② $\overline{BX} \cong \overline{XC}$

③ $AB = AC$ → ④ $\overline{AB} \cong \overline{AC}$ → ⑤ $\angle B \cong \angle C$

→ ⑥ $\overline{AB} \cong \overline{AC}$

→ ⑦ $\triangle ABX \cong \triangle ACX$

→ ⑧ $\angle 1 \cong \angle 2$ → ⑨ $\overline{AX} \perp \overline{BC}$ → ⑩ A is on the ⊥ bisector of $\overline{BC}$

Reasons

① A segmt has one & only 1 midpt.

② Def midpoint

③ Given

④ Def ≅ segments.

⑤ If 2 sides ≅, ∠s opp ≅

⑥ Def ≅ segmts

⑦ SAS ≅ SAS

⑧ CPCTC

⑨ If 2 lines form ≅ adj ∠s → ⊥ lines.

⑩ Def. of ⊥ bisector.

Part II: ① Draw $\overline{AX} \perp \overline{BC}$

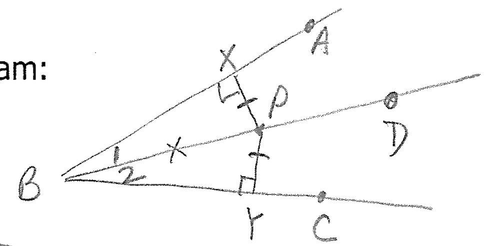
Show that $\overline{BX} \cong \overline{XC}$

If a point is equidistant from the sides of an angle, then it is on the angle bisector.

Given: $\overline{PX} \perp \overrightarrow{BA}$; $\overline{PY} \perp \overrightarrow{BC}$

$PX = PY$

Diagram:



Prove: $\overrightarrow{BP}$ bisects $\measuredangle ABC$

hyleg

① $\overline{PX} \perp \overrightarrow{BA}$ → ② $\measuredangle BXP$ is Rt → ③ $\triangle BXP$ Rt. $\triangle$

$\overline{PY} \perp \overrightarrow{BC}$ → $\measuredangle BYP$ is Rt. → $\triangle BYP$ Rt. $\triangle$

④ $PX = PY$ → ⑤ $\overline{PX} \cong \overline{PY}$

⑥ $\overline{BP} \cong \overline{BP}$

→ ⑦ $\triangle BXP \cong \triangle BYP$ → ⑧ $\measuredangle 1 \cong \measuredangle 2$ → ⑨ $\overrightarrow{BP}$ bisects $\measuredangle ABC$

① Given
② $\perp$ lines form Rt $\measuredangle$s
③ $\triangle$ Rt. Definition
④ Given
⑤ Def $\cong$ segments
⑥ Reflexive Prop.
⑦ hyleg $\cong$ hyleg
⑧ CPCTC
⑨ Def of $\measuredangle$ bisector

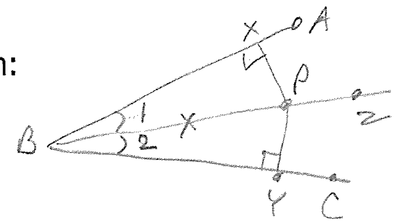
If a point is on an angle bisector, then it is equidistant from the sides of the angle.

Given: $\overrightarrow{BZ}$ bisects $\measuredangle ABC$

P lies on $\overrightarrow{BZ}$

$\overline{PX} \perp \overrightarrow{BA}$; $\overline{PY} \perp \overrightarrow{BC}$

Diagram:



AAS

Prove: $PX = PY$

① $\overline{PX} \perp \overrightarrow{BA}$ → ② $\measuredangle BXP$ Rt $\measuredangle$

$\overline{PY} \perp \overrightarrow{BC}$ → $\measuredangle BYP$ Rt $\measuredangle$

→ ③ $\measuredangle BXP \cong \measuredangle BYP$

④ $\overrightarrow{BZ}$ bisects $\measuredangle ABC$ → ⑤ $\measuredangle 1 \cong \measuredangle 2$

⑥ $\overline{BP} \cong \overline{BP}$

→ ⑦ $\triangle BXP \cong \triangle BYP$ → ⑧ $\overline{PX} \cong \overline{PY}$ → ⑨ $PX = PY$

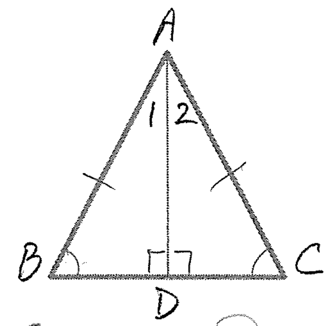
① Given
② $\perp$ lines form Rt $\measuredangle$s
③ All Rt $\measuredangle$s $\cong$
④ Given
⑤ Def. $\measuredangle$ bisector
⑥ Reflexive Prop.
⑦ AAS $\cong$ AAS
⑧ CPCTC
⑨ Def $\cong$ segments

In an isosceles triangle, the altitude to the base is the median to the base and bisects the vertex angle.

Given: Isosceles $\triangle$ ABC with vertex A and $\overline{AD}$ altitude from A.

Prove: $\overline{BD} \cong \overline{CD}$, <1≅ <2

Diagram:



① Isosceles $\triangle ABC$ → ② $\overline{AB} \cong \overline{AC}$ → ③ $\angle B \cong \angle C$

② $\overline{AB} \cong \overline{AC}$

④ $\overline{AD}$ altitude from A → ⑤ $\angle ADB$ is Rt, $\angle ADC$ is Rt → ⑥ $\angle ADB \cong \angle ADC$

→ ⑦ $\triangle ABD \cong \triangle ACD$ AAS

⑧ $\overline{BD} \cong \overline{CD}$

⑨ $\angle 1 \cong \angle 2$

① Given

② If $\triangle$ isos., 2 sides ≅.

③ If 2 sides ≅, $\angle$s opp ≅.

④ Given

⑤ def. of altitude

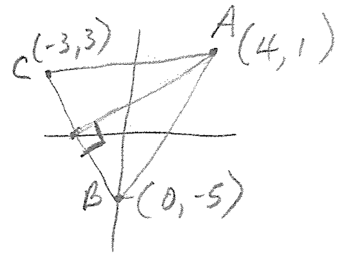
⑥ All Rt $\angle$s ≅.

⑦ AAS ≅ AAS

⑧ CPCTC

⑨ CPCTC

Ex: In ΔABC, A(4,1), B(0,-5), C(-3,3)



a. Find the equation of the altitude from A.

Need slope $\overline{CB}$

$m_{\overline{CB}} = \frac{3+5}{-3} = \frac{8}{-3}$

Slope alt.

$m_{\perp} = \frac{3}{8}$

A (4,1)

$y = mx + b$

$1 = \frac{3}{8}(4) + b$

$1 = \frac{3}{2} + b$

$1 - 1\frac{1}{2} = b$

$-\frac{1}{2} = b$

Altitude

$\boxed{y = \frac{3}{8}x - \frac{1}{2}}$

b. Find the equation of the median from B.

Need midpoint of $\overline{AC}$

$\left(\frac{4-3}{2}, \frac{1+3}{2}\right)$

midpoint $\left(\frac{1}{2}, 2\right)$

Need slope of B → midpoint

$\left(\frac{1}{2}, 2\right)$ $(0, -5)$

$m = \frac{2+5}{\frac{1}{2}} = 7 \times 2 = 14$

$y = mx + b$

$-5 = 14(0) + b$

$-5 = b$

$\boxed{y = 14x - 5}$

* Careful! Perpendicular bisector often does not pass thru vertex.

c. Find the equation of the perpendicular bisector of $\overline{AB}$. (Not from vertex C)

slope of $\overline{AB}$

$m_{\overline{AB}} = \frac{1+5}{4-0} = \frac{6}{4} = \frac{3}{2}$

slope of $\perp$

$m = -\frac{2}{3}$ *

① midpoint of $\overline{AB}$

$\left(\frac{4+0}{2}, \frac{1-5}{2}\right)$

*(2, -2)

y-int

$y = mx + b$

$-2 = -\frac{2}{3}(2) + b$

$-2 + \frac{4}{3} = b$

$-\frac{2}{3} = b$

$\perp$ bisector

$\boxed{y = -\frac{2}{3}x - \frac{2}{3}}$

Write a flow proof for the following.

Given: F is the midpoint of $\overline{BC}$

$\overline{DB} \cong \overline{EC}$; $\overline{DB} \perp \overline{DF}$; $\overline{EC} \perp \overline{EF}$

Prove: $\overline{AF} \perp \overline{BC}$

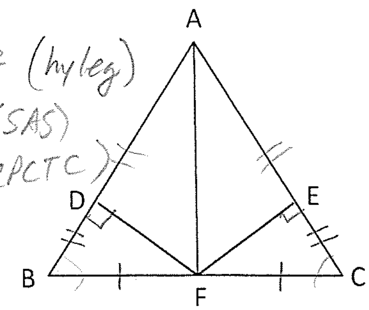
Plan

$\triangle DBF \cong \triangle ECF$ (hyleg)

$\triangle ABF \cong \triangle ACF$ (SAS)

$\measuredangle AFB \cong \measuredangle AFC$ (CPCTC)

$\overline{AF} \perp \overline{BC}$



1 $\overline{DB} \perp \overline{DF}$ → 2 $\measuredangle BDF$ Rt. → 3 $\triangle BDF$ is Rt.

$\overline{EC} \perp \overline{EF}$ → $\measuredangle CEF$ Rt. → $\triangle CEF$ is Rt.

4 F midpt. $\overline{BC}$ → 5 $\overline{BF} \cong \overline{FC}$

6 $\overline{DB} \cong \overline{EC}$

→ 7 $\triangle BDF \cong \triangle CEF$

8 $\measuredangle B \cong \measuredangle C$ → 9 $\overline{AB} \cong \overline{AC}$

8 $\measuredangle B \cong \measuredangle C$

5 $\overline{BF} \cong \overline{FC}$

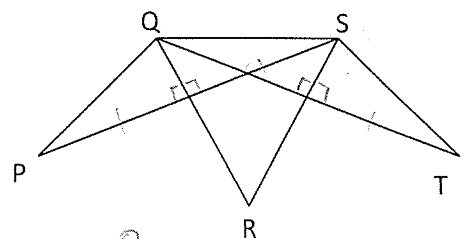
→ 10 $\triangle ABF \cong \triangle ACF$ → 11 $\measuredangle AFB \cong \measuredangle AFC$

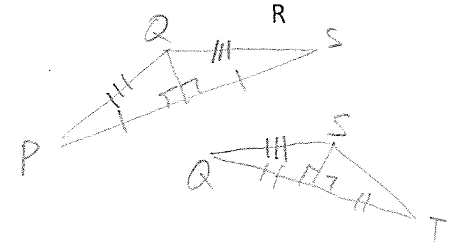
12 $\overline{AF} \perp \overline{BC}$

Given: $\overline{SR}$ is the perpendicular bisector of $\overline{QT}$

$\overline{QR}$ is the perpendicular bisector of $\overline{SP}$

Prove: $\overline{PQ} \cong \overline{TS}$





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