Name ______________________ Class ____________ Date ________

8.3 Medians and Altitudes of Triangles

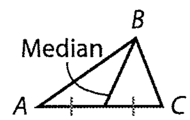
Essential Question: How can you find the balance point or *center of gravity* of a triangle?

Resource Locker

Explore Finding the Balance Point of a Triangle

If a triangle were cut out of a sheet of wood or paper, the triangle could be balanced around exactly one point inside the triangle.

A **median** of a triangle is a segment whose endpoints are a vertex of a triangle and the midpoint of the opposite side.



Every triangle has three distinct medians. You can use construction tools to show that the intersection of the three medians is the balance point of the triangle.

(A) Draw a large triangle on a sheet of construction paper. Label the vertices A, B, and C.

(B) Find the midpoint of the side opposite A, which is $\overline{BC}$. You may use a compass to find points equidistant to B and C and then draw the perpendicular bisector. Or you can use paper folding or a ruler. Write the label X for the midpoint.

(C) Draw a segment to connect A and X. The segment is one of the three medians of the triangle.

(D) Repeat Steps B and C, this time to draw the other two medians of the triangle. Write the label Y for the midpoint of the side opposite point B, and the label Z for the midpoint of the side opposite point C. Write the label P for the intersection of the three medians.

(E) Use a ruler to measure the lengths of each median and the subsegments defined by P in your triangle. Record your measurements in the table.

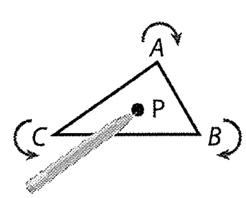
Median $\overline{AX}$:	$AX =$ ______	$AP =$ ______	$PX =$ ______
Median $\overline{BY}$:	$BY =$ ______	$BP =$ ______	$PY =$ ______
Median $\overline{CZ}$:	$CZ =$ ______	$CP =$ ______	$PZ =$ ______

(F) What pattern do you observe in the measurements?

(G) Let AX be the length of any median of a triangle from a vertex A, and let P be the intersection of the three medians. Write an equation to describe the relationship between AP and PX.

(H) Let AX be the length of any median of a triangle from a vertex A, and let P be the intersection of the three medians. Write an equation to show the relationship between AX and AP.

(I) Cut out the triangle, and then punch a very small hole through P. Stick a pencil point through the hole, and then try to spin the triangle around the pencil point. How easily does it spin? Repeat this step with another point in the triangle, and compare the results.



Reflect

1. Why is "balance point" a descriptive name for point P, the intersection of the three medians?

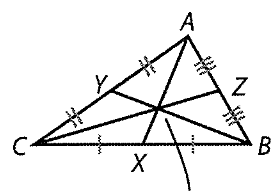
2. **Discussion** By definition, median $\overline{AX}$ intersects $\triangle ABC$ at points A and X. Could it intersect the triangle at a third point? Explain why or why not.

Explain 1 Using the Centroid Theorem

The intersection of the three medians of a triangle is the *centroid* of the triangle. The centroid is always inside the triangle and divides each median by the same ratio.

Centroid Theorem

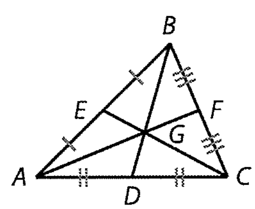
The centroid theorem states that the **centroid** of a triangle is located $\frac{2}{3}$ of the distance from each vertex to the midpoint of the opposite side.



$AP = \frac{2}{3}AX$ $\quad BP = \frac{2}{3}BY$ $\quad CP = \frac{2}{3}CZ$

Example 1 **Use the Centroid Theorem to find the length.**

$AF = 9$, and $CE = 7.2$



(A) AG

Centroid Theorem $\quad AG = \frac{2}{3}AF$

Substitute 9 for AF. $\quad AG = \frac{2}{3}(9)$

Simplify. $\quad AG = 6$

(B) GE

Centroid Theorem	$CG = \frac{2}{3}$ ______
Substitute for the given value.	$CG = \frac{2}{3}$ ______
Simplify.	$CG =$ ______
Segment Addition Postulate	$CG +$ ______ $= CE$
Subtraction Property of Equality	$GE = CE -$ ______
Substitute for the value of CG.	$GE = 7.2 -$ ______
Simplify.	$GE =$ ______

Reflect

3. To find the centroid of a triangle, how many medians of the triangle must you construct?

____

4. Compare the lengths of $\overline{CG}$ and $\overline{GE}$ in Part B. What do you notice?

____

5. **Make a Conjecture** The three medians of $\triangle FGH$ divide the triangle into six smaller triangles. Is it possible for the six smaller triangles to be congruent to one another? If yes, under what conditions?

____

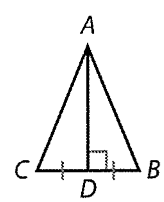
____

Your Turn

6. Vertex L is 8 units from the centroid of $\triangle LMN$. Find the length of the median that has one endpoint at L.

7. Let P be the centroid of $\triangle STU$, and let $\overline{SW}$ be a median of $\triangle STU$. If $SW = 18$, find SP and PW.

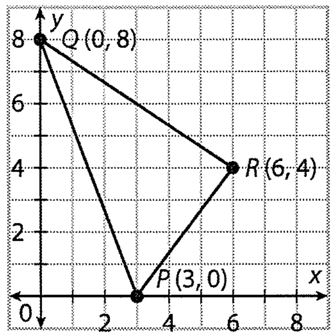
8. In $\triangle ABC$, the median $\overline{AD}$ is perpendicular to $\overline{BC}$. If $AD = 21$ feet, describe the position of the centroid of the triangle.



Explain 2 Finding the Intersection of Medians of a Triangle

When a triangle is plotted on the coordinate plane, the medians can be graphed and the location of the centroid can be identified.

Example 2 **Find the coordinates of the centroid of the triangle shown on the coordinate plane.**



Analyze Information

What does the problem ask you to find? ____________

What information does the graph provide that will help you find the answer?

____

Formulate a Plan

The centroid is the ________ of the medians of the triangle. Begin by calculating the ________ of one side of the triangle. Then draw a line to connect that point to a ________. You need to draw only ________ medians to find the centroid.

Solve

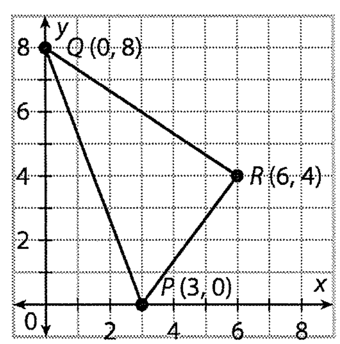
Find and plot midpoints.

Let M be the midpoint of $\overline{QR}$.

$$M = \left(\frac{0+6}{2}, \frac{8+4}{2}\right) = \_\_\_\_$$

Let N be the midpoint of $\overline{QP}$.

$$N = \left(\frac{0+3}{2}, \frac{8+0}{2}\right) = \_\_\_\_$$



Draw the medians and identify equations.

Draw a segment to connect M and ________.

The segment is a median and is described by the equation ________.

Draw a segment to connect N and ________.

The segment is also a median and is described by the equation ________.

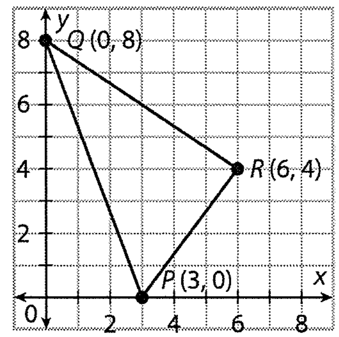
Find the centroid.

Identify the intersection of the two medians, which is (________). Label it C.

Justify and Evaluate

The answer seems reasonable because it is positioned in the middle of the triangle. To check it, find the midpoint of $\overline{RP}$, which is ______. Label the midpoint L, and draw the third median, which is ______. The slope of the third median is $\frac{2-8}{4.5-0} = -\frac{4}{3}$, and the equation that describes it is $y = -\frac{4}{3}x +$ ______. It intersects the other two medians at (______), which confirms C as the centroid.

You can also apply the Centroid Theorem to check your answer.



$RC = \frac{2}{3}RN$

$RN =$ ______

$RC =$ ______

Substitute values into the first equation:

$3 = \frac{2}{3}$ ______

The equality is true, which confirms the answer.

Your Turn

Find the centroid of the triangles with the given vertices. Show your work and check your answer.

9. $P(-1, 7)$, $Q(9, 5)$, $R(4, 3)$

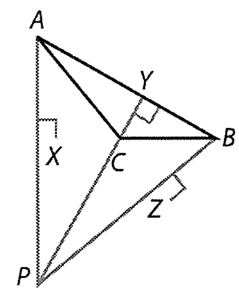
10. $A(-6, 0)$, $B(0, 12)$, $C(6, 0)$

Explain 3 Finding the Orthocenter of a Triangle

Like the centroid, the *orthocenter* is a point that characterizes a triangle. This point involves the *altitudes* of the triangle rather than the medians.

An **altitude** of a triangle is a perpendicular segment from a vertex to the line containing the opposite side. Every triangle has three altitudes. An altitude can be inside, outside, or on the triangle.

In the diagram of $\triangle ABC$, the three altitudes are $\overline{AX}$, $\overline{BZ}$, and $\overline{CY}$. Notice that two of the altitudes are outside the triangle.



The length of an altitude is often called the height of a triangle.

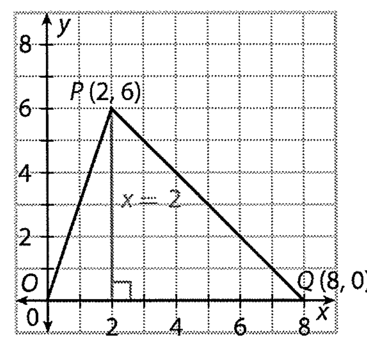
The **orthocenter** of a triangle is the intersection (or point of concurrency) of the lines that contain the altitudes. Like the altitudes themselves, the orthocenter may be inside, outside, or on the triangle. Notice that the lines containing the altitudes are concurrent at P. The orthocenter of this triangle is P.

Example 3 **Find the orthocenter of the triangle by graphing the perpendicular lines to the sides of the triangle.**

(A) **Step 1** Draw the triangle. Choose one vertex and then find and graph the equation of the line containing the altitude from that vertex.

Triangle with vertices $O(0, 0)$, $P(2, 6)$, and $Q(8, 0)$

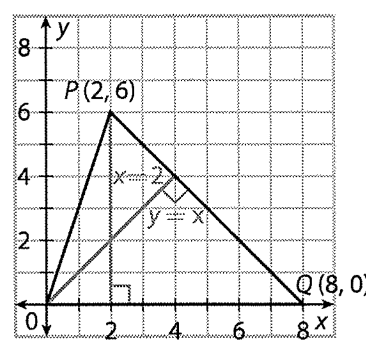
Choose P. The side opposite P is $\overline{OQ}$, which is horizontal, so the altitude is vertical. The altitude is a segment of the line $x = 2$.



Step 2 Repeat Step 1 with a second vertex.

Choose O, the origin. The altitude that contains O is perpendicular to $\overline{PQ}$. Calculate the slope of $\overline{PQ}$ as $\frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 0}{2 - 8} = -1$.

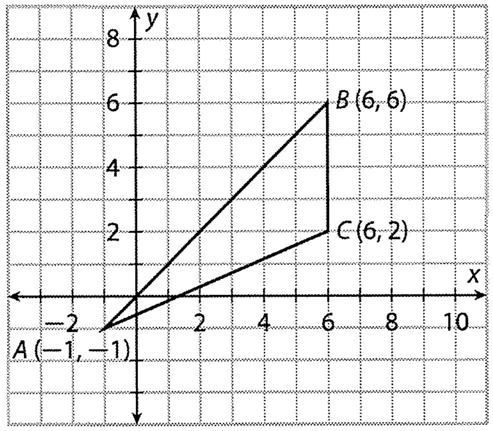
Since the slope of the altitude is the opposite reciprocal of the slope of $\overline{PQ}$, the slope of the altitude is 1. The altitude is a segment of the line that passes through the origin and has a slope of 1. The equation of the line is $y = x$.



Step 3 Find the intersection of the two lines.

The orthocenter is the intersection of the two lines that contain the altitudes. The lines $x = 2$ and $y = x$ intersect at $(2, 2)$, which is the orthocenter.

(B)



Step 1 Find the altitude that contains vertex A.

Because $\overline{BC}$ is vertical, the altitude through A is a ________ segment. The equation of the line that contains the segment is $y =$ ____ . Draw this line.

Step 2 Find the altitude that contains vertex C.

First, calculate the slope of $\overline{AB}$. The slope is $\frac{6 - \_\_\_}{6 - \_\_\_}$, which equals ____ .

The slope of the altitude to $\overline{AB}$ is the ________________ of 1, which is -1.

Use the point-slope form to find the equation of the line that has a slope of -1 and passes through ____:

$y -$ ____ $= -1(x -$ ____$)$, which simplifies to $y = -x + 8$.

Draw this line.

Step 3 Find the intersection of the two lines.

$y = -1$

$y = -x + 8$

Substitute for y:

____ $= -x + 8$

$x =$ ____

The orthocenter is at (________).

Reflect

11. Could the orthocenter of a triangle be concurrent with one of its vertices? If yes, provide an example. If not, explain why not.

____

____

12. An altitude is defined to be a perpendicular segment from a vertex to the line containing the opposite side. Why are the words "the line containing" important in this definition?

____

____

Your Turn

Find the orthocenter for the triangles described by each set of vertices.

13. $Q(4, -3)$, $R(8, 5)$, $S(8, -8)$

14. $K(2, -2)$, $L(4, 6)$, $M(8, -2)$

Elaborate

15. Could the centroid of a triangle be coincident with the orthocenter? If so, give an example.

16. Describe or sketch an example in which the orthocenter P of $\triangle ABC$ is far away from the triangle. That is, PA, PB, and PC are each greater than the length of any side of the triangle.

17. A sculptor is assembling triangle-shaped pieces into a mobile. Describe circumstances when the sculptor would need to identify the centroid and orthocenter of each triangle.

18. **Essential Question Check-In** How can you find the centroid, or balance point, of a triangle?

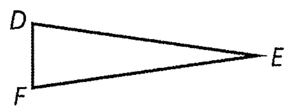
Evaluate: Homework and Practice

Personal Math Trainer

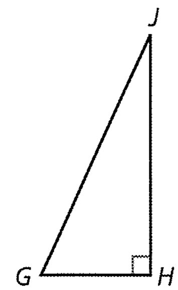
- Online Homework
- Hints and Help
- Extra Practice

Use a compass and a straightedge to draw the medians and identify the centroid of the triangle. Label the centroid *P*.

1.



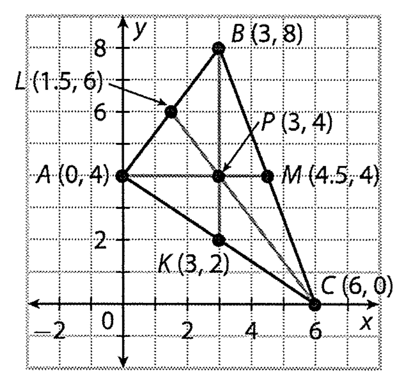
2.



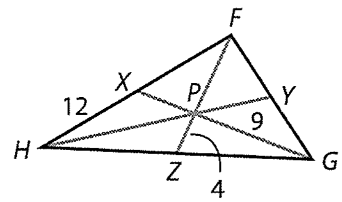
3. **Critique Reasoning** Paul draws $\triangle ABC$ and the medians from vertices A and B. He finds that the medians intersect at a point, and he labels this point X. Paul claims that point X lies outside $\triangle ABC$. Do you think this is possible? Explain.

4. For $\triangle ABC$ and its medians, match the segment on the left with its length.

A. AM	______	1.5
B. AP	______	2
C. PM	______	2.5
D. BK	______	3
E. BP	______	4
F. PK	______	4.5
G. CL	______	5
H. CP	______	6
I. PL	______	7.5



The diagram shows △*FGH*, its medians, centroid *P*, and the lengths of some of the subsegments. Apply the Centroid Theorem to find other lengths.

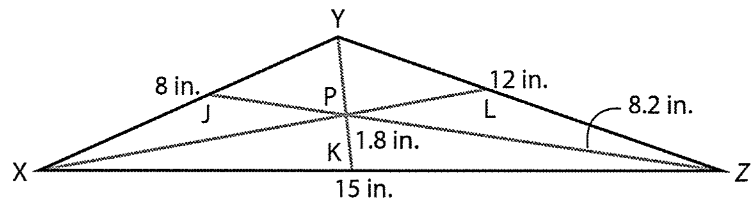


5. *FH*

6. *PF*

7. *GX*

The diagram shows △*XYZ*, which has side lengths of 8 inches, 12 inches, and 15 inches. The diagram also shows the medians, centroid *P*, and the lengths of some of the subsegments. Apply the Centroid Theorem to find other lengths.



8. *LY*

9. *KY*

10. *ZJ*

11. The diagram shows △*ABC*, its medians, centroid *P*, and the lengths of some of the subsegments as expressions of variables *x* and *y*. Apply the Centroid Theorem to solve for the variables and to find other lengths.

a. x

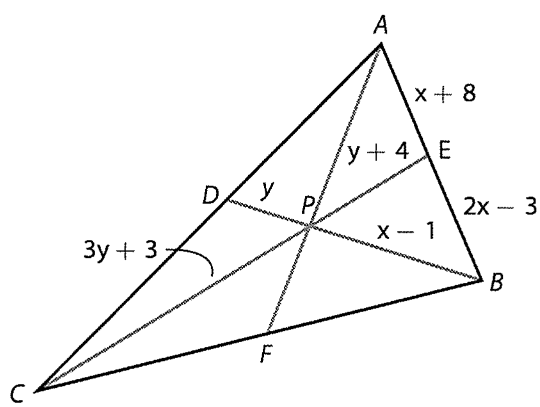
b. y

c. *BP*

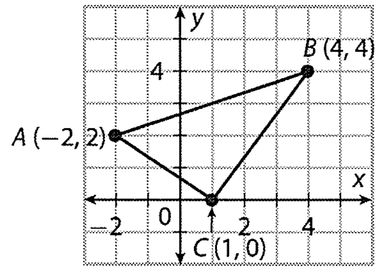
d. *BD*

e. *CP*

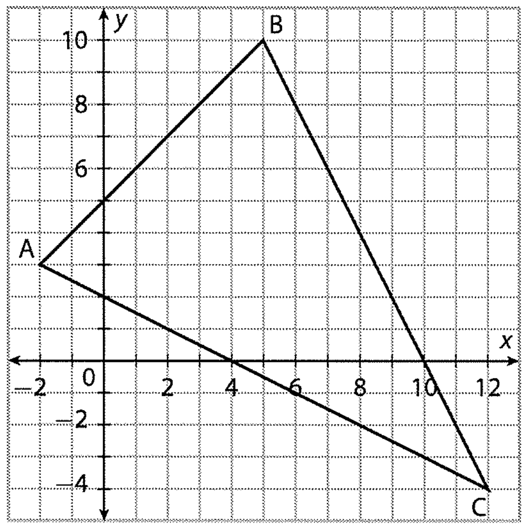
f. *PE*



12. Draw the medians from A to $\overline{BC}$ and from C to $\overline{AB}$.



The vertices of a triangle are $A(-2, 3)$, $B(5, 10)$, and $C(12, -4)$. Find the coordinates or equations for each feature of the triangle.



13. the coordinates of the midpoint of $\overline{AC}$

14. the coordinates of the midpoint of $\overline{BC}$

15. the equation of the line that contains the median through point B

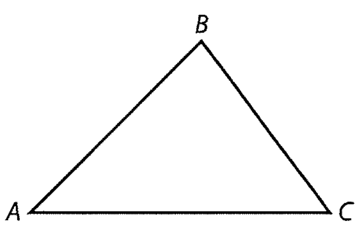
16. the equation of the line that contains the median through point A

17. the coordinates of the intersection of the two medians

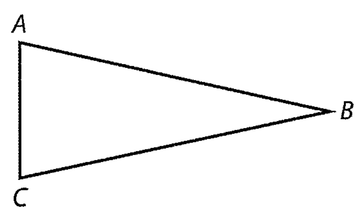
18. the coordinates of the center of balance of the triangle

For each triangle, draw the three altitudes and find the orthocenter. Label it P.

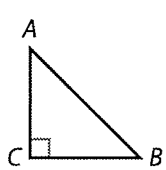
19.



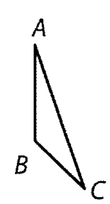
20.



21.



22.



Find the orthocenter of each triangle with the given vertices.

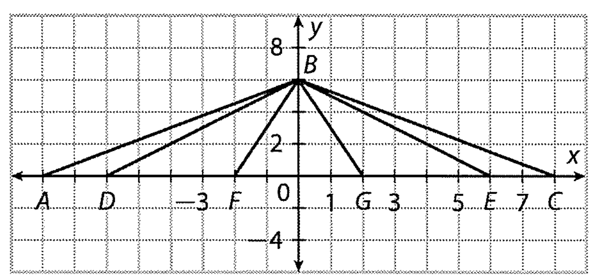
23. $A(2, 2)$, $B(2, 10)$, $C(4, 2)$

24. $A(2, 5)$, $B(10, -3)$, $C(4, 5)$

25. $A(9, 3)$, $B(9, -1)$, $C(6, 0)$

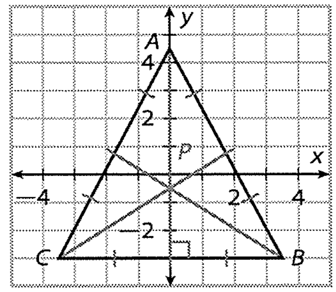
H.O.T. Focus on Higher Order Thinking

26. Draw Conclusions Triangles *ABC*, *DBE*, and *FBG* are all symmetric about the *y*-axis. Show that each triangle has the same centroid. What are the coordinates of the centroid?

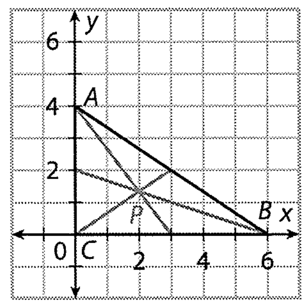


27. Analyze Relationships Triangle *ABC* is plotted on the coordinate plane. $\overline{AB}$ is horizontal, meaning it is parallel to the *x*-axis. $\overline{BC}$ is vertical, meaning it is parallel to the *y*-axis. Based on this information, can you determine the location of the orthocenter? Explain.

28. **What if?** The equilateral triangle shown here has its orthocenter and centroid on the y-axis. Suppose the triangle is stretched by moving A up the y-axis, while keeping B and C stationary. Describe and compare the changes to the centroid and the orthocenter of the triangle.

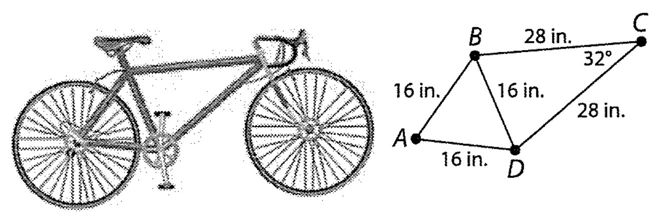


29. **What if?** The diagram shows right triangle ABC on the coordinate plane, and it shows the three medians and centroid P. How does the position of the centroid change when the triangle is stretched by moving B to the right along the x-axis, and keeping A and C stationary? How does the orthocenter change?



Lesson Performance Task

A bicycle frame consists of two adjacent triangles. The diagram shows some of the dimensions of the two triangles that make up the frame.



Answer these questions about the bicycle frame $ABCD$. Justify each of your answers.

a. Find the measures of all the angles in the frame.

b. Copy the figure on a piece of paper. Then find the center of gravity of each triangle.

c. Estimate the center of gravity of the entire frame and show it on your diagram.

d. Explain how you could modify the frame to lower its center of gravity and improve stability.